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THE STABILITY OF ARCHES IN THE LAGRANGIAN APPROACH

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Abstract—This article concerns the bridge arch whose axis is the funicular curve of its own weight. All the hypotheses can be quite general, the structures may even be unsymmetrical, the ends may have an angular cedibility; the loads may be of any kind and the moment of inertia may vary according to a generic law. The arch is reduced to a set of rigid bars and elastic springs, as is often the case when dealing with straight bars. The advantage of this procedure is that it converges from below, and it can operate along with finite or similar elements. Comparisons with some of the results of the literature were carried out. There are some differences between our results and those concerning various steep arches, which is also explained.

NOTATION

$\mathbf{B}$ matrix of the potential energy
C_i specific elastica of order i
$\mathbf{e}$ eigenvector
E Young's modulus
f height of the arch
k elastic flexural deformability
$\mathbf{K}$ matrix of the strain energy
h_i height of the generic spring i
I moment of inertia
l length of the arch
L strain energy
n number of Lagrangian coordinates
$\mathbf{q}$ vector of forces
P potential energy
s_i length of the generic bar i
$\mathbf{v}$ vector of the vertical displacements
$\mathbf{w}$ vector of the horizontal displacements
α angle between the horizontal axis and the tangent line to the centroidal axis
γ nondimensional coefficient in the Timoshenko-Gere formula
φ vector of Lagrangian coordinates

INTRODUCTION

The object of this study is to examine the critical multipliers and the critical deformed shapes of a bridge arch. It is common knowledge that an arch's axis is the funicular curve of its own weight. Weight, and hence axis, are as general as possible, even nonsymmetric; e.g. the ends can be at two different levels, as is often the case. This also applies to the moment of inertia I of the section. Comparative examples have been supplied, taking into account the results of the literature. An arch with a parabolic axis is assumed, whose moment of inertia is either constant or variable according to cos α or $\cos^3 \alpha$.

In the literature, the circular arch with a radial load has been thoroughly investigated, but unfortunately this case does not apply to bridge structures.

Several well-known methods can be applied to solve this problem. Essentially they fall into two groups: In the first, the structure is considered a monodimensional continuum; in the second, the structure is reduced to a discrete system. The first case leads to differential or integral equations of deformed shapes, and is solved by investigating the spectrum of the eigenvalues. In the second case, fictitious systems are dealt with when the whole spectrum is not easily found (Ritz methods, finite difference methods etc.). Alternatively, we can consider fictitious systems, where the arch is reduced to a set of rigid nodes linked together by elastic bars (finite elements methods).

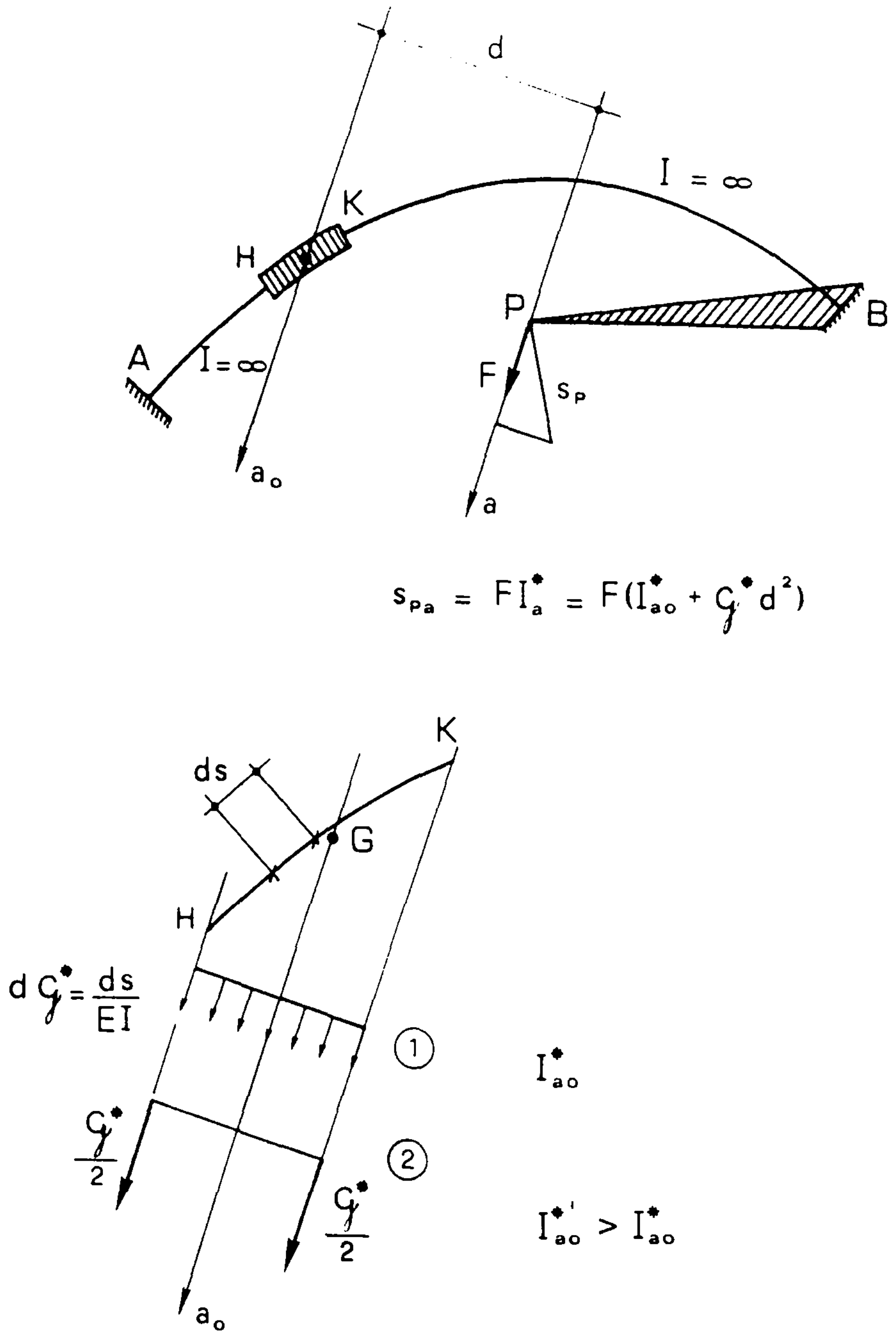

Fig. 1.

The authors believe the discretization according to Lagrange has not been applied to the calculation of the critical loads of the arch bridge, even if it is often used for straight bars, probably because of the difficulty in expressing second-order terms of the potential energy of the applied forces.

In this article, the authors suggest adopting precisely this method; they believe it can be usefully applied along with the existing methods, even if only for comparison. This method can be used, in fact, by means of simple programming, requiring a small memory. Because of its convergence from below, it is particularly useful if considered with the finite elements method. Disregarding axial deformations, Lagrange discretization consists, as already known, of subdividing the structure into a number t of bars such that every bar can be considered a prismatic one, and of assimilating the elastic weight s_i/EI_i of every bar to a pair of elastic weights $s_i/2EI_i$, concentrated at the ends of the bar. In this way, the idealized structure results in an orthodox holonomic system, according to the dictates of classical mechanics. It is formed by rigid elements linked together by means of punctual constraints. The reactions that the constraints transfer to the bars are conservative forces, as for the applied forces, and their potential is the strain energy.

The $n = t - 2$ rotations φ_i of the bars, from the second to the second last, take on the role of Lagrangian coordinates. Thus, we can see that the procedure approximates to the eigenvalue with increasing values when the number n of the springs is increasing (the same convergence does not require any mathematical proof, obviously because the idealized structure approaches the real one when n increases). Although an exact proof of this fact is desirable, at least an intuitive justification can be given.

Let us consider an arch with its left side fixed and its right side (Fig. 1) free, with a general force F applied to a point P belonging to the free section. The component s_F of P displacement in the direction of F calculated on the idealized structure is greater than the same component calculated on the real structure. This is obvious if the contribution of the single segment is examined. Hence, the strain energy $L = \frac{1}{2}Fs_F$, linked to F, is greatest when calculated on the idealized structure.

But the strain energy is also given by $\frac{1}{2}\int MM'\ ds/EI$, and its increase is equivalent to a decrease of I; this causes a decrease of critical multipliers.

Generally, the potential energy P of the external forces is expandable in a Taylor series of the Lagrangian coordinates

$$P = P^{(1)} + P^{(2)} + \cdots + P^{(i)}$$

where $P^{(i)}$ is the term of order i.

If $P^{(1)} \neq 0$, the forces are termed "transversal forces," if $P^{(1)} = 0$, "axial forces." The arch axis is a funicular curve of its own weight. Hence, its own weight is, in the Lagrange reduction, an axial load (see below). Therefore, when the displacements are small, the total potential energy $E_t = L + P$ is a quadratic function of the φ_i.

This is a classical Eulerian case. Hence, it is a linear eigenvalue problem.

2. THE REDUCTION OF THE ARCH TO A RIGID ELASTIC SYSTEM

The structure is divided (Fig. 2) into a finite number t of rigid bars, linked together by springs placed in the centroid of the assemblage section.

In the generic spring i, the elastic flexural deformability of the i segment included between the midpoint of the precedent bar and the midpoint of the subsequent one is

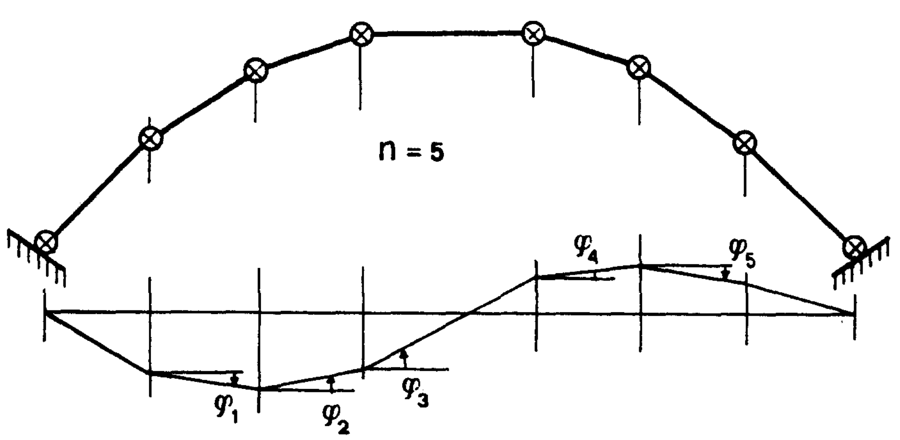

Fig. 2.

concentrated. Hence,

$$M = k_i \Delta \varphi_i. \tag{1}$$

From the equality

$$\Delta\varphi = \frac{M}{k_i} = Mc_i = \frac{M}{2E}\left(\frac{s_i}{I_i} + \frac{s_{i+1}}{I_{i+1}}\right),$$

we have

$$c_i = \frac{1}{2E}\left(\frac{s_i}{I_i} + \frac{s_{i+1}}{I_{i+1}}\right). \tag{2}$$

At the ends A and B, we obtain

$$c_A = \frac{s_0}{2EI_0}, \qquad c_B = \frac{s_{n+1}}{2EI_{n+1}}, \tag{3}$$

and at the crown

$$c_C = \frac{1}{2E}\left(\frac{s_{n/2}}{I_{n/2}} + \frac{s_{n/2+1}}{I_{n/2+1}}\right). \tag{4}$$

Giving to (4), to (3) or to both a value equal to ∞, we have arches with one, two or three hinges, respectively.

Considering the "small displacement hypothesis" (Euler hypothesis) valid, the strain energy is reduced to

$$L = \tfrac{1}{2}\boldsymbol{\varphi}^T\mathbf{K}\boldsymbol{\varphi}. \tag{5}$$

$\boldsymbol{\varphi}^T\mathbf{K}\boldsymbol{\varphi}$ is a positive definite quadratic form, and $\mathbf{K}$ is the matrix of this form.

If $\varphi_i = 1$ and $\varphi_j = 0$, $\forall j \neq i$ ("specific" elastica C_i of order i), we have

$$2L = k_{ii}. \tag{6}$$

Hence, k_{ii} is twice the strain energy joined to $\varphi_i = 1$ and $\varphi_j = 0$, $\forall j \neq i$. If $\varphi_i = 1$ and $\varphi_j = 1$ and all the other coordinates are zero, we have

$$L = \tfrac{1}{2}(k_{ii} + k_{jj} + 2k_{ij}). \tag{7}$$

Hence, k_{ij} is the mutual energy between the two elastica C_i and C_j. The potential energy P is generally the sum of a linear form and of a quadratic one:

$$P = -\boldsymbol{\varphi}^T\mathbf{q} - \frac{\lambda}{2}\boldsymbol{\varphi}^T\mathbf{B}\boldsymbol{\varphi}. \tag{8}$$

With respect to the elastica C_i, we have

$$P = -Q_i - \frac{\lambda}{2}B_{ii},$$

where Q_i is the first-order work of the applied loads relative to C_i, and λB_{ii} is twice the second-order work.

With regard to C_i and C_j, we have

$$P = -Q_i - Q_j - \frac{\lambda}{2} B_{ii} - \frac{\lambda}{2} B_{jj} - \lambda B_{ij}.$$

Hence, λB_{ij} is the second-order "added" work of the applied forces relative to C_i and C_j.

The term "mutual" is not used, so that confusion with "mutual work" between two sets of applied forces is ruled out.

The equilibrium condition

$$\frac{\partial E_r}{\partial \varphi_i} = 0$$

leads to the system of n equations:

$$\mathbf{K}\boldsymbol{\varphi} - \lambda \mathbf{B}\boldsymbol{\varphi} = \mathbf{q}. \tag{9}$$

As already stated, transversal forces are associated with $\mathbf{q} \neq 0$; axial forces are associated with $\mathbf{q} = 0$ and $\mathbf{B} \neq 0$. When the forces are axial, the system is homogeneous:

$$\mathbf{K}\boldsymbol{\varphi} - \lambda \mathbf{B}\boldsymbol{\varphi} = 0. \tag{10}$$

When λ is such that

$$\det | \mathbf{K} - \lambda \mathbf{B} | = 0, \tag{11}$$

system (9) has no definite solutions, and system (10) has a nontrivial solution (defined with a degree of freedom).

From system (10), we have

$$\bar{\mathbf{K}}\boldsymbol{\varphi} - \lambda \boldsymbol{\varphi} = 0, \tag{12}$$

where $\bar{\mathbf{K}} = \mathbf{B}^{-1}\mathbf{K}$. Hence, (11) takes the form

$$\det | \bar{\mathbf{K}} - \lambda \mathbf{I} | = 0. \tag{13}$$

$\bar{\mathbf{K}}$ is a symmetrizable matrix. Thus, (13) has n real eigenvalues. Since the two matrices are positive definite, the eigenvalues λ_i are positive. The n distinct eigenvectors $e_i, \ldots, e_n$ (buckling modes) are orthogonal, i.e.

$$\mathbf{e}_h^T \mathbf{K} \mathbf{e}_i = 0, \qquad \text{if } i \neq h \tag{14}$$

and also

$$\mathbf{e}_h^T \mathbf{B} \mathbf{e}_i = 0, \qquad \text{if } i \neq h. \tag{15}$$

3. THE CONSTRUCTION OF THE STRAIN ENERGY MATRIX

The arch is reduced to a rigid elastic system by dividing it into t bars (t is assumed to be even); $h(z)$ is the height of the point at the abscissa z, and it is assumed greater than zero if the point is above the z-axis. The number c of the springs is $t + 1$; the Lagrangian freedoms are $n = c - 3 = t - 2$. In the generic spring i, the elastic flexibility c_i of the segment included between the midpoint of the bar i and the midpoint of the bar $i + 1$ is concentrated; in (2) we must assume

$$s_i = \int_{z_i}^{z_{i+1}} \left[1 + \left(\frac{dh}{dz} \right)^2 \right]^{1/2} dz. \tag{16}$$

At the end A, the flexibility c_A of the segment included between A and the midpoint of the first bar is concentrated; the flexibility c_A^* of the constraint A must be added to c_A. At the end B, the flexibility c_B of the segment included between B and the midpoint of the last bar is concentrated; the flexibility c_B^* of the constraint B must be added to c_B.

The input data of the problem are (Fig. 3) the span l, the rise f, the elastic modulus E and the two flexibilities c_A^* and c_B^*. Furthermore, the $n + 1$ values h, z and F corresponding to the springs from 0 to $n + 1$ and the $n + 3$ values c_i corresponding to the springs from A to B must be given. The values F_i are obtained by decomposing the load of the generic bar into two concentrated forces acting at the ends of the bar.

It is suitable to consider separately the values h_0, z_0, h_{n+1}, F_{n+1} and to arrange the remaining values h_i, z_i and F_i into n-dimensional vectors.

The rotations φ_i of the n bars $(1, \ldots, n)$ take on the role of Lagrangian coordinates. The rotations φ_A and φ_B of the two lateral bars are homogeneous linear functions of φ_i. Freeing the arch at B and assuming $\varphi_i \neq 0$ and $\varphi_j = 0$, $\forall j \neq i$, the displacements at B are (Fig. 4):

$$v_B = -\varphi_A t_0 - \varphi_i t_i - \varphi_B t_{n+2} \tag{17}$$

$$w_B = -\varphi_A h_0 + \varphi_i (h_i - h_{i-1}) - \varphi_B (h_B - h_{n+2}). \tag{18}$$

The boundary conditions $v_B = w_B = 0$ yield, if the ends are at the same level,

$$\varphi_A + \varphi_B = -\varphi_i \frac{t_i}{t_0} \tag{19}$$

$$\varphi_A - \varphi_B = -\varphi_i \frac{h_i - h_{i-1}}{h_0}, \tag{20}$$

which result in

$$\varphi_A = -\frac{1}{2} \varphi_i \frac{t_i}{t_0} - \frac{1}{2} \varphi_i \frac{h_i - h_{i-1}}{h_0} \tag{21}$$

$$\varphi_B = -\frac{1}{2} \varphi_i \frac{t_i}{t_0} + \frac{1}{2} \varphi_i \frac{h_i - h_{i-1}}{h_0}. \tag{22}$$

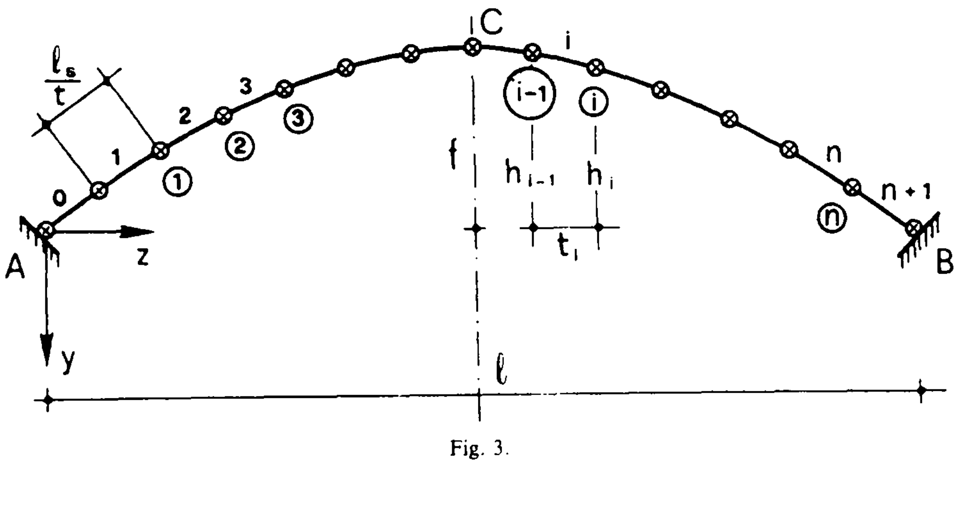

Fig. 3.

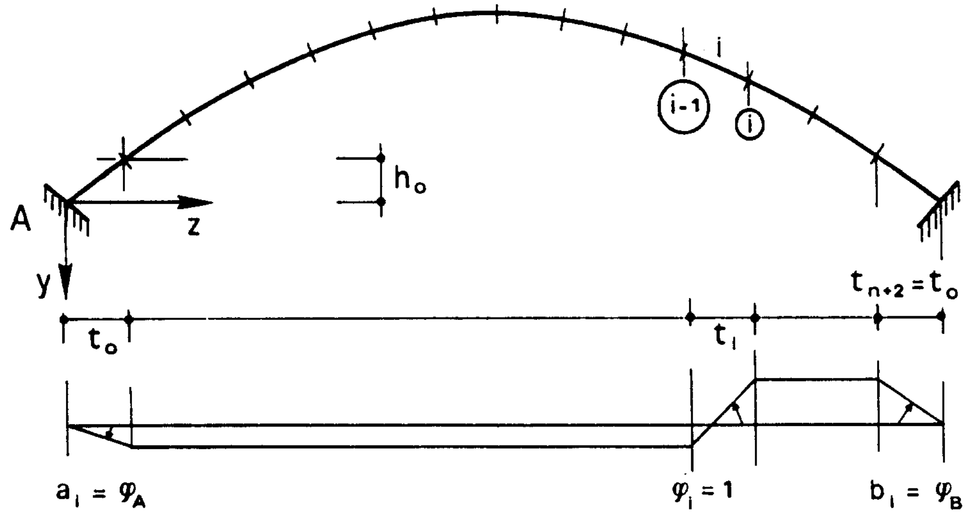

Fig. 4.

In general,

$$\begin{aligned} \varphi_A &= \sum_i \varphi_i a_i = \boldsymbol{\varphi}^T \mathbf{a} \\ \varphi_B &= \sum_i \varphi_i b_i = \boldsymbol{\varphi}^T \mathbf{b}. \end{aligned} \tag{23}$$

The strain energy L is the sum of

1. the energy related to the springs included between 1 and $n - 1$:

$$\frac{1}{2} \sum_{i=1}^{n-1} k_i(\varphi_{i+1} - \varphi_i)^2.$$

2. the energy related to the springs θ and n:

$$\tfrac{1}{2}k_0(\varphi_1 - \varphi_A)^2 + \tfrac{1}{2}k_n(\varphi_B - \varphi_n)^2$$

3. the energy related to the springs A and B:

$$\tfrac{1}{2}(k_A\varphi_A^2 + k_B\varphi_B^2),$$

where $1/k_A = c_A + c_A^*$, $1/k_B = c_B + c_B^*$. Therefore, we have

$$L = \frac{1}{2}[\varphi_A^2(k_A + k_0) + \varphi_B^2(k_n + k_B)] - \frac{1}{2}(2\varphi_A\varphi_1 k_0 + 2\varphi_B\varphi_n k_n) + \frac{1}{2}\sum_{i=1}^{n-1} k_i(\varphi_{i+1} - \varphi_i)^2 + \frac{1}{2}(k_0\varphi_1^2 + k_n\varphi_n^2).$$

From (23),

$$\begin{aligned} \varphi_A^2 &= \boldsymbol{\varphi}^T \mathbf{a}\mathbf{a}^T \boldsymbol{\varphi} \\ \varphi_B^2 &= \boldsymbol{\varphi}^T \mathbf{b}\mathbf{b}^T \boldsymbol{\varphi}, \end{aligned} \tag{24}$$

where $\mathbf{a}^T = (a_1, a_2, \ldots, a_n)$ and $\mathbf{b}^T = (b_1, b_2, \ldots, b_n)$.

On the other hand, we have

$$\sum_{i=1}^{n-1} k_i(\varphi_{i+1} - \varphi_i)^2 + k_0\varphi_1^2 + k_n\varphi_n^2$$

$$\begin{aligned} &= k_0\varphi_1^2 + k_1\varphi_1^2 + k_1\varphi_2^2 - 2k_1\varphi_1\varphi_2 \\ &\quad + k_2\varphi_2^2 + k_2\varphi_3^2 - 2k_2\varphi_2\varphi_3 \\ &\quad + \cdot \qquad \cdot \qquad \cdot \\ &\quad + k_{n-1}\varphi_{n-1}^2 + k_{n-1}\varphi_n^2 - 2k_{n-1}\varphi_{n-1}\varphi_n + k_n\varphi_n^2, \\ &= \boldsymbol{\varphi}^T k' \boldsymbol{\varphi} \end{aligned} \tag{25}$$

where

$$\mathbf{K}' = \left|\begin{array}{ccccc:ccc} k'_{11} & k'_{12} & \theta & \theta & \theta & \theta & \theta & \theta \\ k'_{21} & k'_{22} & k'_{23} & \theta & \theta & \theta & \theta & \theta \\ \theta & k'_{12} & k'_{33} & k'_{34} & \theta & \theta & \theta & \theta \\ \hdashline \theta & \theta & \theta & \theta & \theta & k'_{n-1,n-2} & k'_{n-1,n-1} & k'_{n-1,n} \\ \theta & \theta & \theta & \theta & \theta & \theta & k'_{n,n-1} & k'_{n,n} \end{array}\right|$$

with

$$\begin{aligned} k'_{i-1,i} &= -k_{i-1} \\ k'_{i,i} &= k_{i-1} + k_i \\ k'_{i+1,i} &= -k_i. \end{aligned}$$

Terms involving $\varphi_A\varphi_1$, and $\varphi_B\varphi_n$ can be written as

$$\begin{aligned} \varphi_A\varphi_1 &= \varphi_1\boldsymbol{\varphi}^T\mathbf{a} \\ \varphi_B\varphi_n &= \varphi_n\boldsymbol{\varphi}^T\mathbf{b}, \end{aligned} \tag{26}$$

i.e.

$$\begin{aligned} \varphi_A\varphi_1 &= \boldsymbol{\varphi}^T\mathbf{A}'\boldsymbol{\varphi} \\ \varphi_B\varphi_1 &= \boldsymbol{\varphi}^T\mathbf{B}'\boldsymbol{\varphi}, \end{aligned} \tag{27}$$

where

$$\mathbf{A}' = \frac{1}{2}\begin{vmatrix} 2a_1 & a_2 & \cdots & a_n \\ a_2 & \theta & \cdots & \theta \\ \cdot & \cdot & \cdots & \cdot \\ a_n & \theta & \cdots & \theta \end{vmatrix}; \qquad \mathbf{B}' = \frac{1}{2}\begin{vmatrix} 2b_1 & b_2 & \cdots & b_n \\ b_2 & \theta & \cdots & \theta \\ \cdot & \cdot & \cdots & \cdot \\ b_n & \theta & \cdots & \theta \end{vmatrix}.$$

In conclusion, the strain energy is given by

$$L = \tfrac{1}{2}\boldsymbol{\varphi}^T\mathbf{K}\boldsymbol{\varphi}, \tag{28}$$

where

$$\mathbf{K} = \mathbf{K}' + (k_0 + k_A)\mathbf{a}\mathbf{a}^T + (k_n + k_B)\mathbf{b}\mathbf{b}^T - 2k_0\mathbf{A}' - 2k_n\mathbf{B}'. \tag{29}$$

4. THE CONSTRUCTION OF THE MATRIX B OF THE POTENTIAL ENERGY

A rigid bar PQ rotating at an angle φ, with P fixed, yields (Fig. 5):

$$\begin{aligned} v_Q &= r \sin \alpha - r \sin(\alpha + \varphi) \\ &= r \sin \alpha - r \sin \alpha \cos \varphi - r \cos \alpha \sin \varphi \\ &= r \sin \alpha(1 - 1 + \varphi^2) - r\varphi \cos \alpha \\ &= -a\varphi - b\frac{\varphi^2}{2}. \\ w_Q &= r \cos(\alpha + \varphi) - r \cos \alpha \\ &= -r\varphi \sin \alpha - r \cos \alpha \frac{\varphi^2}{2} \\ &= b\varphi - a\frac{\varphi^2}{2}. \end{aligned}$$

We have, therefore,

$$\begin{aligned} v_Q^{(2)} &= -b\frac{\varphi^2}{2} \\ w_Q^{(2)} &= -a\frac{\varphi^2}{2}. \end{aligned} \tag{30}$$

$v_Q^{(2)}$ is always negative. $w_Q^{(2)}$ is positive if Q is higher than $\mathbf{B}(b < 0)$; otherwise it is negative.

The calculation of $\mathbf{B}$ is carried out in two steps. First, displacements (30) are taken into account, computing them beginning from the left end. Every bar i rotates, with respect to bar $i - 1$, with hinge $i - 1$ fixed. Therefore, the subsequent second-order

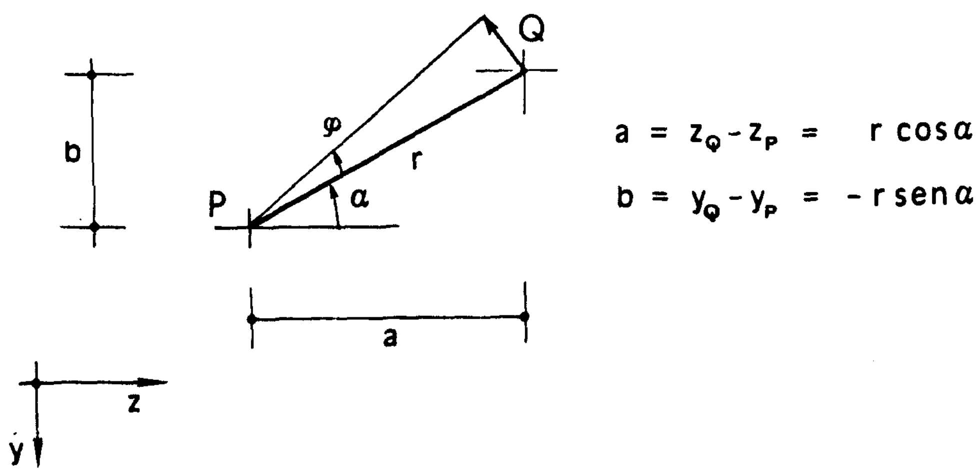

Fig. 5.

displacements $\nu_i'^{(2)}$ are obtained:

$$
\begin{aligned}
\nu_0'^{(2)} &= \frac{h_0}{2}\varphi_A^2\\
\nu_1'^{(2)} &= \nu_0'^{(2)} + \frac{h_1 - h_0}{2}\varphi_1^2\\
&\cdot \quad \cdot \quad \cdot\\
\nu_i'^{(2)} &= \nu_{i-1}'^{(2)} + \frac{h_i - h_{i-1}}{2}\varphi_i^2\\
&\cdot \quad \cdot \quad \cdot\\
\nu_{n+1}'^{(2)} &= \nu_n'^{(2)} + \frac{h_B - h_n}{2}\varphi_B^2.
\end{aligned}
$$

The potential energy P' of the applied forces linked to the displacements $\nu_i'^{(2)}$ is

$$
\begin{aligned}
P' = &-\frac{1}{2}\varphi_A^2 h_0 F_0 + \left(-\frac{1}{2}\varphi_A^2 h_0 - \frac{h_1 - h_0}{2}\varphi_1^2\right)F_1\\
&+ \left(-\frac{1}{2}\varphi_A^2 h_0 - \frac{h_1 - h_0}{2}\varphi_1^2 - \frac{h_2 - h_1}{2}\varphi_2^2\right)F_2\\
&+ \quad \cdot \quad \cdot\\
&+ \frac{1}{2}\varphi_B^2\left(-\frac{h_B - h_n}{2}\right)F_{n+1}\\
= &-\varphi_A^2\frac{h_0}{2}\left(F_0 + \sum_{i=1}^{n} F_i\right) - \sum_{i=1}^{n}\left(\varphi_i^2\frac{h_i - h_{i-1}}{2}\sum_{k=i}^{n} F_k\right)\\
&- \varphi_B^2\frac{h_B - h_n}{2}F_{n+1},
\end{aligned}
$$

i.e. (24):

$$
\begin{aligned}
P' = &-\boldsymbol{\varphi}^T\mathbf{a}\mathbf{a}^T\boldsymbol{\varphi}\frac{h_0}{2}\left(F_0 + \sum_{i=1}^{n} F_i\right)\\
&- \boldsymbol{\varphi}^T\mathbf{b}\mathbf{b}^T\boldsymbol{\varphi}\frac{h_B - h_n}{2}F_{n+1} - \boldsymbol{\varphi}^T\mathbf{C}\boldsymbol{\varphi}.
\end{aligned}
$$

C is a diagonal matrix:

$$
C_{ii} = \frac{h_i - h_{i-1}}{2}\sum_{k=i}^{n} F_k. \tag{31}
$$

But P' does not represent the whole potential energy P. We must add the potential energy arising from the fact that the displacements $v_i'^{(2)}$ and $w_i'^{(2)}$ do not respect the boundary conditions. (See Fig. 6, in which $v_B \neq 0$ and $w_B \neq 0$.) To meet these conditions, it is necessary to give other displacements $v_i''^{(2)}$, to which displacements $-v_B$ and $-w_B$ must be linked. These are second-order displacements, because $-v_B$ and $-w_B$ are second order.

The $v_i''^{(2)}$ must be recalculated from the undeformed configuration (small displacements hypothesis). Hence, the applied forces F_i and the reactions V_B and H_B are in equilibrium on it. The reactive couples that the springs transfer to the bars are equal to zero, because the axis is a funicular curve. According to the Lagrange theorem, we

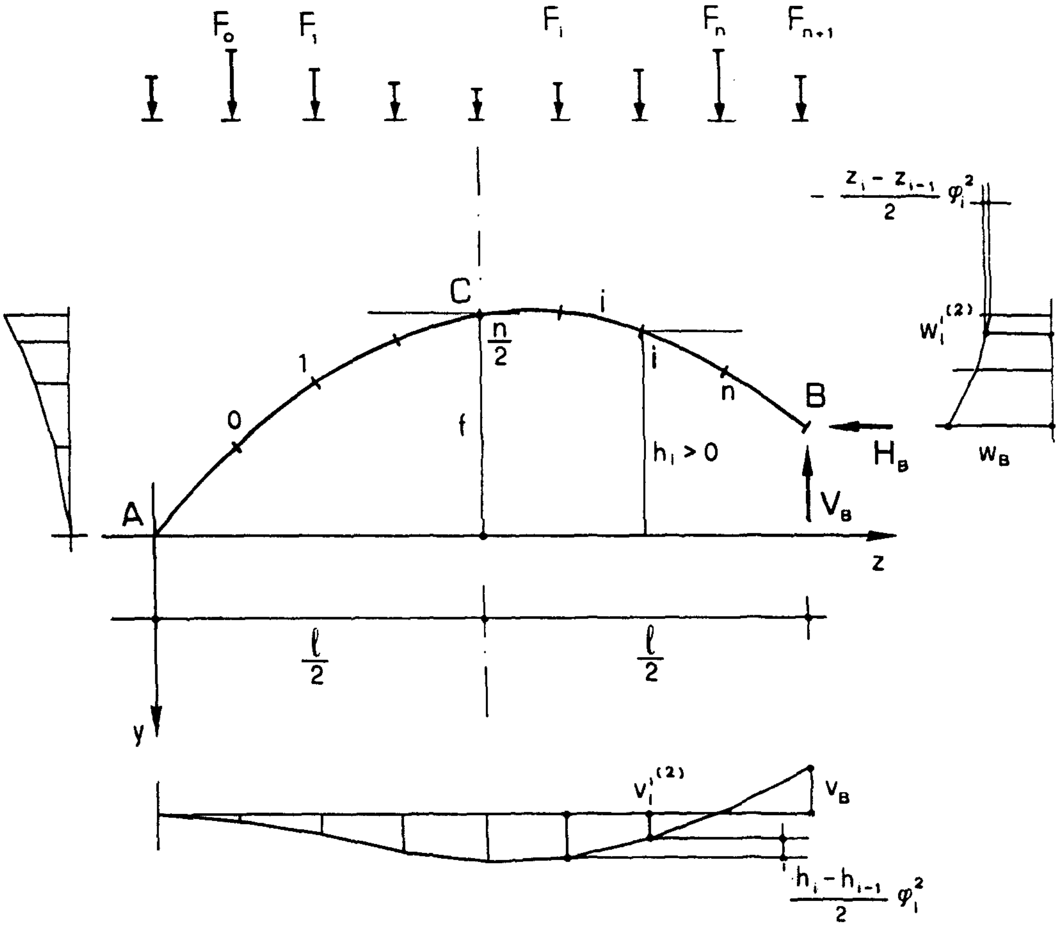

Fig. 6.

have

$$\sum_{i=0}^{n+1} F_i v_i''^{(2)} - V_B v_B - H_B w_B = 0, \tag{32}$$

from which

$$P'' = V_B v_B - H_B w_B. \tag{33}$$

It is fundamental to observe that term (33) is defined by v_B and w_B, whatever the values of $v_i''^{(2)}$. The significance of this observation arises from the fact that ∞ values of $v_i''^{(2)}$ exist, corresponding to the displacements $-v_B$ and $-w_B$.

v_B is given by

$$\begin{aligned} v_B &= \frac{1}{2} h_0 \varphi_A^2 + \sum_{i=1}^{n} \frac{h_i - h_{i-1}}{2} \varphi_i^2 \\ &\quad + \frac{1}{2} \varphi_B^2 (h_B - h_n) \\ &= -\boldsymbol{\varphi}^T \mathbf{a}\mathbf{a}^T \boldsymbol{\varphi} \frac{h_0}{2} + \boldsymbol{\varphi}^T \mathbf{b}\mathbf{b}^T \boldsymbol{\varphi} \frac{h_B - h_n}{2} + \boldsymbol{\varphi}^T \mathbf{D} \boldsymbol{\varphi}, \end{aligned} \tag{34}$$

where $\mathbf{D}$ is a diagonal matrix:

$$D_{ii} = \frac{h_i - h_{i-1}}{2}. \tag{35}$$

w_B is given by

$$w_B = -\frac{1}{2} z_0 \varphi_A^2 - \sum_{i=1}^{n} \frac{z_i - z_{i-1}}{2} \varphi_i^2 - \frac{l - z_n}{2} \varphi_B^2$$
$$= -\varphi^T \mathbf{a}\mathbf{a}^T \varphi \frac{z_0}{2} - \varphi^T \mathbf{b}\mathbf{b}^T \varphi \frac{l - z_n}{2} - \varphi^T \mathbf{E} \varphi, \tag{36}$$

where **E** is a diagonal matrix:

$$E_{ii} = \frac{z_i - z_{i-1}}{2}. \tag{37}$$

The reactions V_B and H_B are shown by the following equilibrium equations at A and C (Fig. 7):

$$V_B l + H_B h_B + F_0 z_0 + \sum_{i=1}^{n} F_i z_i + F_{n+1} l = 0$$
$$V_B \frac{l}{2} - H_B (f - h_B) + \sum_{i=n/2+1}^{n} F_i \left(z_i - \frac{l}{2} \right) + F_{n+1} \frac{l}{2} = 0. \tag{38}$$

Hence, (33) gives

$$P'' = -\varphi^T \mathbf{a}\mathbf{a}^T \varphi \left(V_B \frac{h_0}{2} - H_B \frac{z_0}{2} \right)$$
$$- \varphi^T \mathbf{b}\mathbf{b}^T \varphi \left(V_B \frac{h_B - h_n}{2} - H_B \frac{l - z_n}{2} \right)$$
$$- \varphi^T \mathbf{D} \varphi V_B + \varphi^T \mathbf{E} \varphi H_B. \tag{39}$$

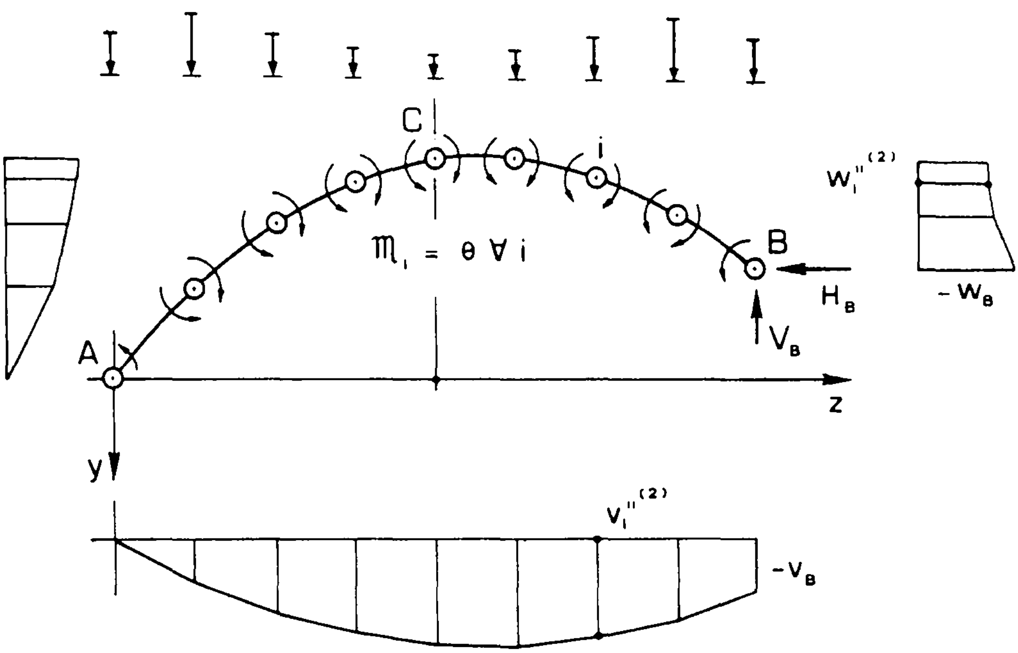

Fig. 7.

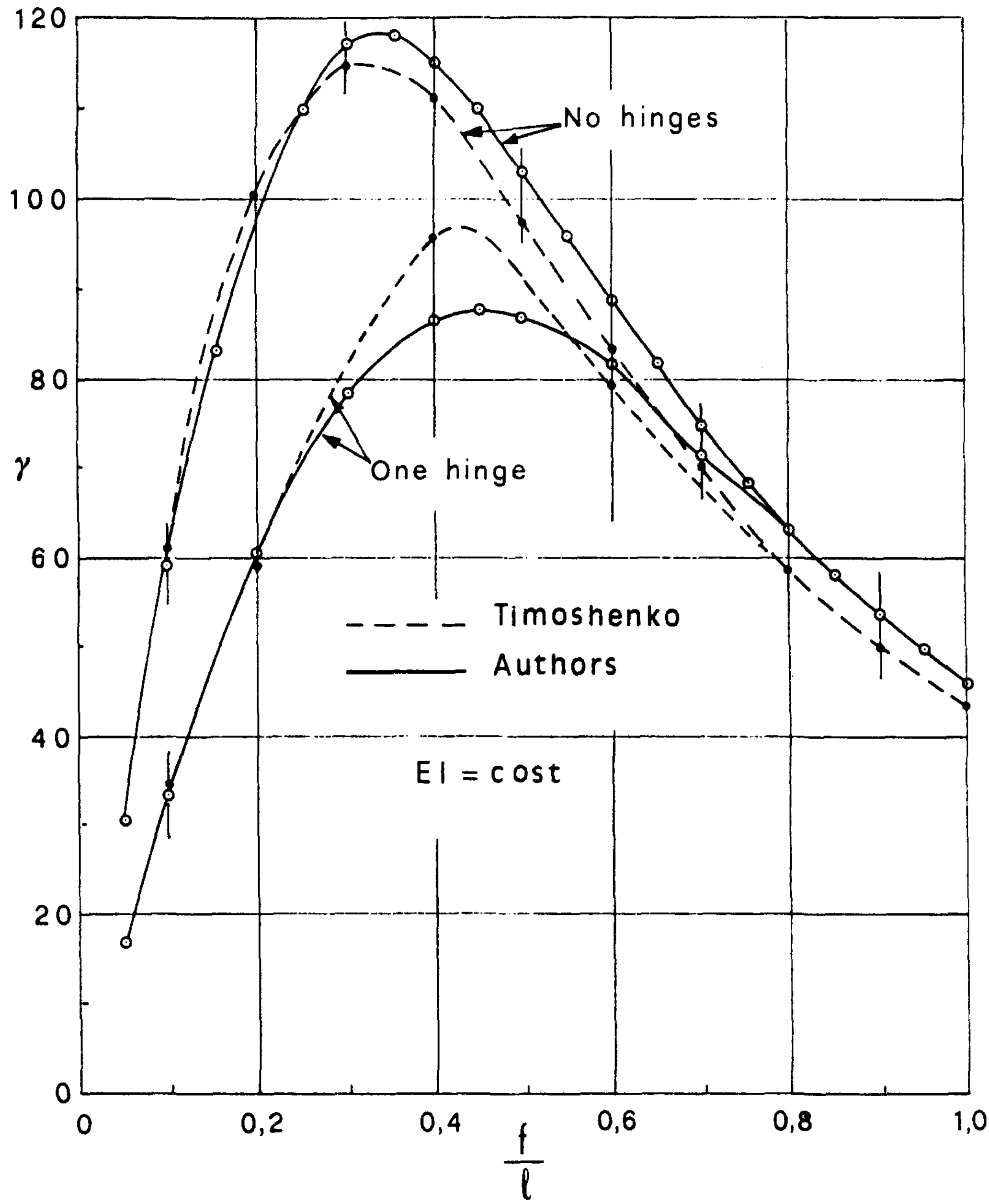

Fig. 8.

Finally, we have

$$P = \tfrac{1}{2}\boldsymbol{\varphi}^T\mathbf{B}\boldsymbol{\varphi}, \tag{40}$$

where

$$\begin{aligned}\frac{1}{2}\,\mathbf{B} = \mathbf{a}\mathbf{a}^T&\left[h_0\left(F_0 + \sum_{i=1}^{n} F_i\right) + V_B h_0 - H_B z_0\right]\\ &+ \mathbf{b}\mathbf{b}^T[(h_B - h_n)F_{n+1} + V_B(h_B - h_n) - H_B(l - z_n)]\\ &+ \mathbf{C} + \mathbf{D}V_B - \mathbf{E}H_B.\end{aligned} \tag{41}$$

If the ends are at the same level and q is a uniform load, the axis is given by

$$h(z) = \frac{4f}{l^2} z(l - z).$$

If the t bars have a constant horizontal projection, we have

$$z_{i+1} - z_i = \frac{l}{t},$$

and B is reduced to

$$\mathbf{B} = q\left[(\mathbf{a}\mathbf{a}^T + \mathbf{b}\mathbf{b}^T)\left(\frac{l^3}{8ft} + \frac{h_0}{2} l \frac{t-1}{t}\right) + \mathbf{C}'\right], \tag{42}$$

where $\mathbf{C}'$ is a diagonal matrix:

$$C'_{ii} = \frac{l^3}{8ft} + (h_i - h_{i-1})\left(l\frac{t+1}{2t} - z_i\right). \tag{43}$$

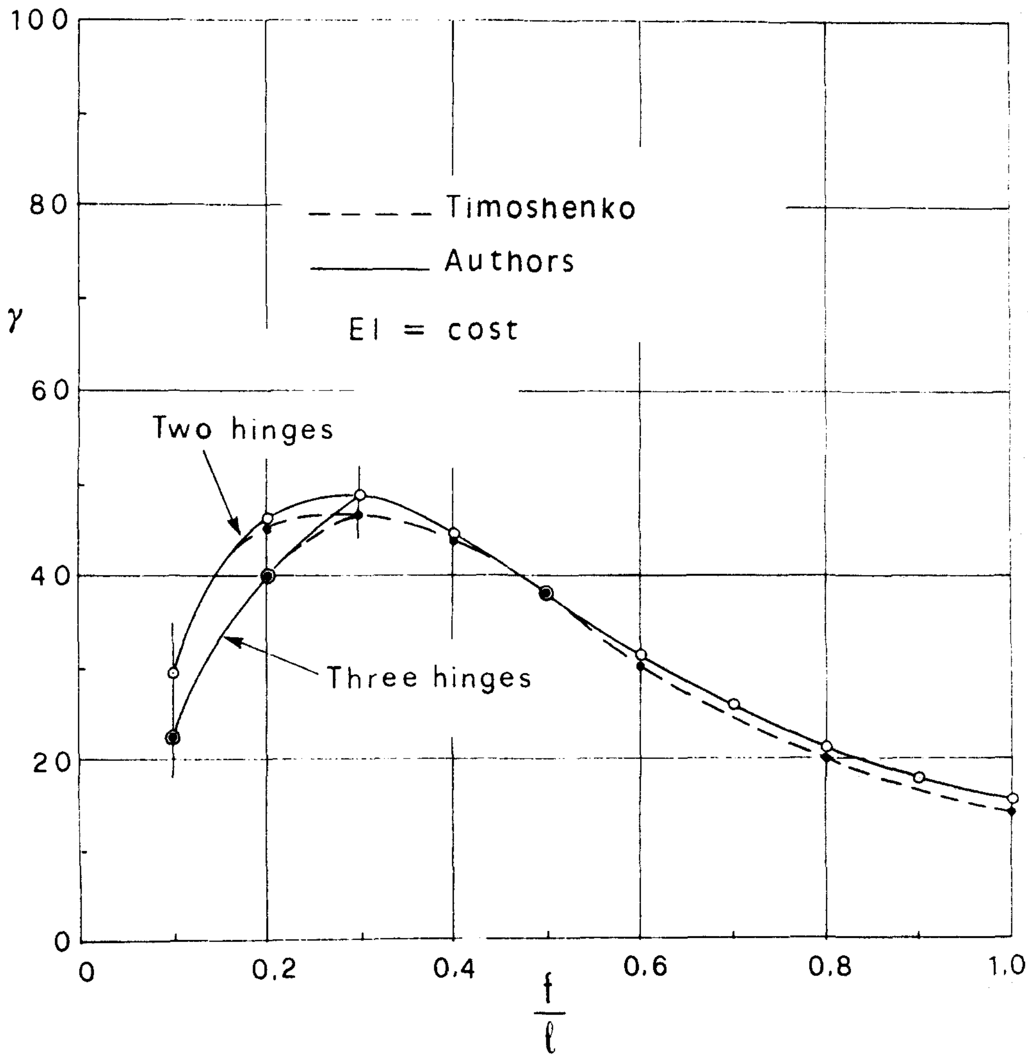

Fig. 9.

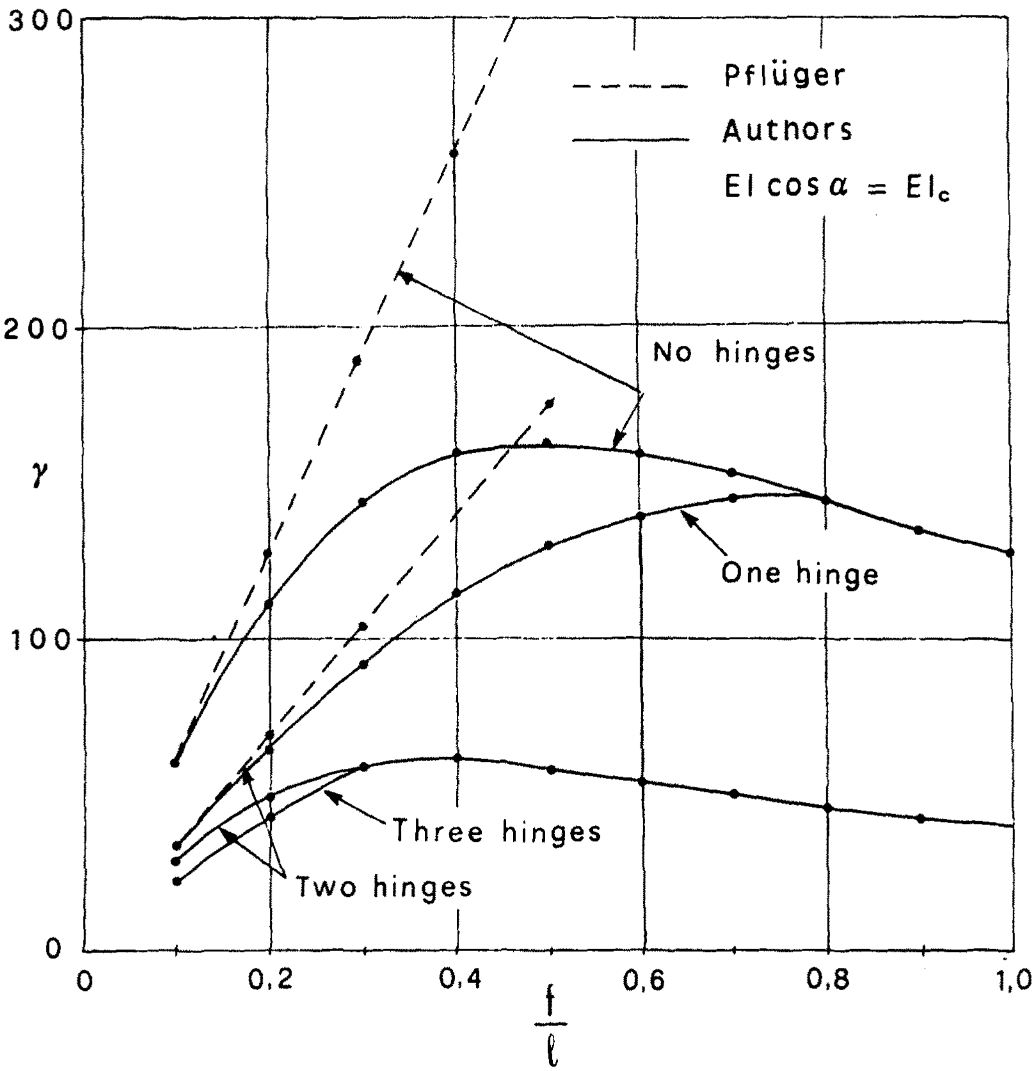

Fig. 10.

5. COMPARISON WITH THE RESULTS FROM THE LITERATURE

To carry out the comparisons, we used a parabolic arch. Almost all the results from the literature refer to this type of arch. The graphs and the diagrams of the following give the critical load against the ratio f/l.

The classical formula (valid for every boundary condition),

$$q_c = \gamma \frac{EI}{l^3},$$

has been used; it was also used by Timoshenko and Gere[1] and Pflüger[2] for EI = const., $EI \cos \alpha$ = const., $EI \cos^3 \alpha$ = const.

The case of a clamped arch and the case of the one-hinge arch are dealt with in Fig. 8. We assume EI = const. The Timoshenko-Gere curves are dashed. The calculation was executed by dividing the arch into 20 bars. The difference between the values obtained with 10 bars and those obtained with 20 bars is nearly 5%.

In Fig. 9, two- and three-hinge arches are considered, again with EI = const.

In Fig. 10, the cross-section that varies according to $EI \cos \alpha$ = const. is dealt with.

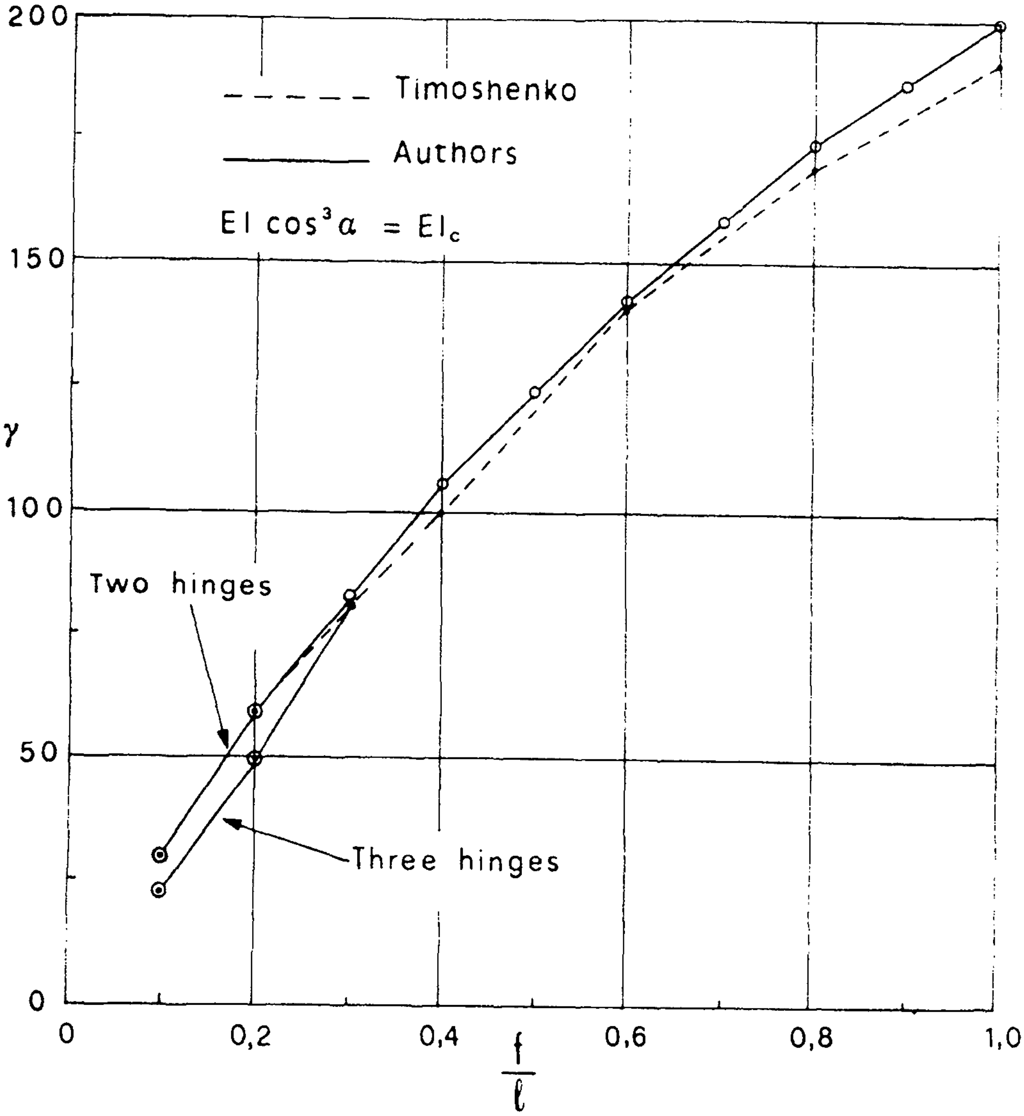

Fig. 11.

Table 1. Nondimensional coefficient γ for EI = const.

	Clamped		One hinge		Two hinges		Three hinges	
f/l	Timoshenko and Gere [1]	Present authors	Timoshenko and Gere [1]	Present authors	Timoshenko and Gere [1]	Present authors	Timoshenko and Gere [1]	Present authors
0.1	60.7	59.7	33.8	33.2	28.5	28.8	22.5	22.5
0.2	101	101	59	60.5	45.4	45.7	39.6	39.9
0.3	115	117.8	—	78.5	46.5	49.1	–	–
0.4	111	115.3	96	86.8	43.9	44.6	–	–
0.5	97.4	103	—	87	38.4	38	–	–
0.6	83.8	89	80	81.9	30.5	31.4	–	–
0.7	71	75	—	74.3	—	26	–	–
0.8	59.1	63	–	–	20	21.4	–	–
0.9	50	53.9	–	–	—	18	–	–
1	43.7	46	–	–	14.1	15.1	–	–

Table 2. Nondimensional coefficient γ for $EI \cos \alpha$ = const.

f/l	Clamped		One hinge		Two hinges		Three hinges	
	Pflüger [2]	Present authors	Pflüger [2]	Present authors	Pflüger [2]	Present authors	Pflüger [2]	Present authors
0.1	60	61.2	—	34	30	29.4	—	23.1
0.2	127	111.4	—	65.8	62	50.1	—	43.6
0.3	190	144.3	—	93.5	90	59.6	—	+
0.4	257	160.5	—	115.5	125	61.4	—	+
0.5	320	164	—	131.2	160	59.3	—	+
0.6	—	161	—	141.1	—	55.5	—	+
0.7	—	154	—	145.8	—	51.5	—	+
0.8	—	145.3	—	+	—	47.4	—	+
0.9	—	136.5	—	+	—	43.5	—	+
1	—	128	—	+	—	40.4	—	+

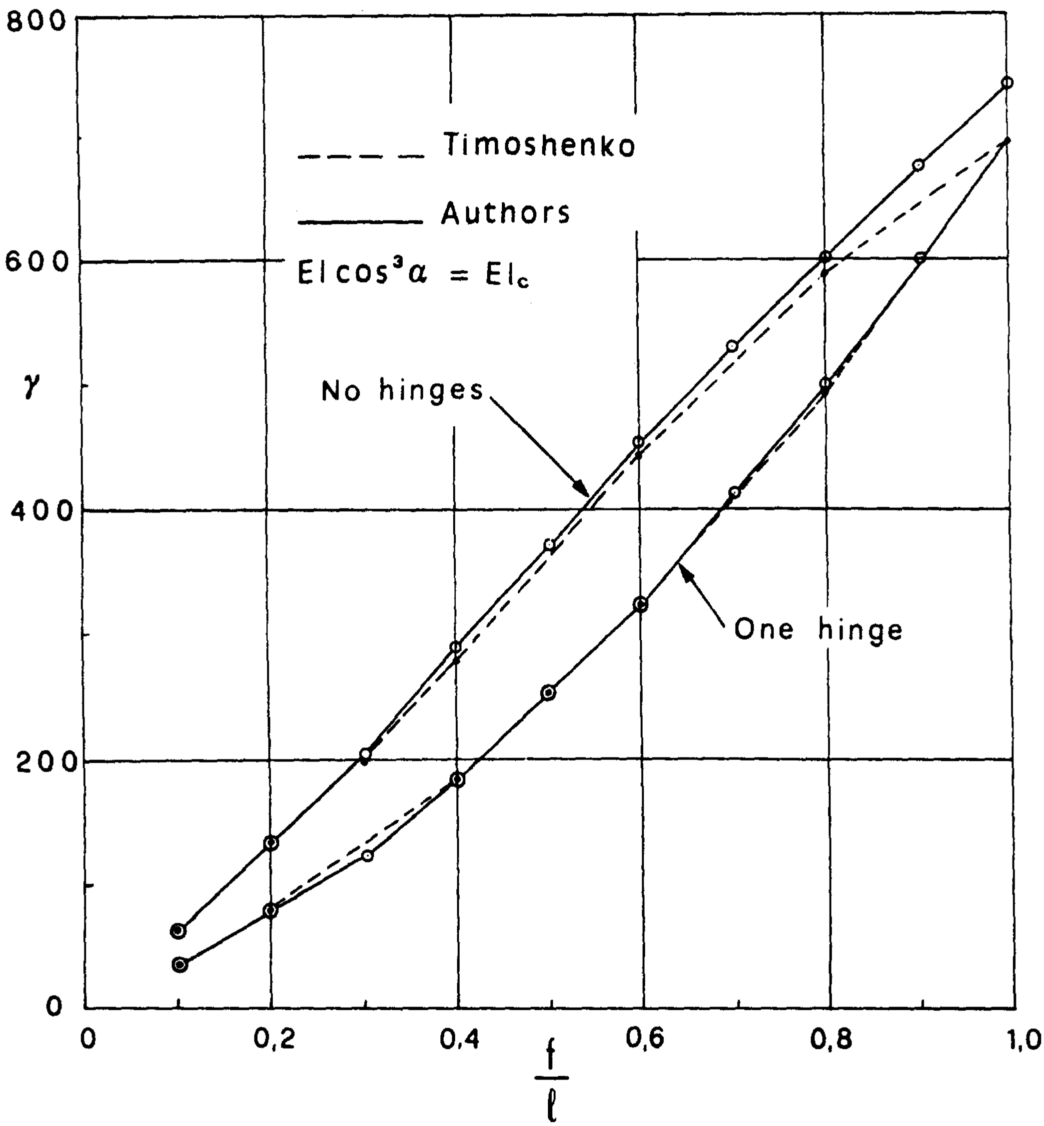

Fig. 12.

In Figs. 11 and 12, cases of the section varying according to $EI \cos^3 \alpha$ = const. are illustrated. The inertia increases from the crown to the sides, and the critical load increases monotonically with the ratio f/l.

Finally, in Fig. 13, the arches for which the section varies according to $EI/\cos \alpha$ = const. are considered.

The inertia of the cross-section diminishes from the crown to the ends: It is the so-called Boussiron-Vallette arch. The similarity between this curve (for the clamped arch) and the curve of the two-hinge arch with EI = const. can be easily seen. Numerical comparisons are reported in Tables 1–4.

NOTE

It is not possible to give a direct explanation of the differences between the results of Pflüger and ours, because in Pflüger's book no specific theoretic indication on the arches is reported, but rather various abaci.

We should also say that Pflüger's results coincide with those of Dishinger (Bauingenieur 1947). On the contrary, Timoshenko and Gere do not report results for the law $I \cos \alpha = I_c$, for which Pflüger gives his results. It is possible, nevertheless, to give an indirect explanation. If the hypothesis of smallness of displacements is made, the equations of the arch elastica are

$$\frac{d^2 v}{dz^2} = -\frac{M}{EI \cos \alpha} \tag{44}$$

$$\frac{d^2 w}{dz^2} = -\frac{M}{EI \sin \alpha}, \tag{45}$$

$v(z)$ and $w(z)$ are the vertical and horizontal components of the displacement, and they must obey this equation:

$$\frac{d^2 w}{dy^2} = \frac{d^2 v}{dz^2} \frac{1}{tg\, \alpha}.$$

When f/l is sufficiently small (but snap-buckling must be avoided), we can consider only eqn (44). In this case we have

$$M = Hv.$$

Hence,

$$\frac{d^2 v}{dz^2} + \frac{Hv}{EI \cos \alpha} = 0. \tag{46}$$

If we consider, as Pflüger does,

$$EI \cos \alpha = EI_c = \text{const.},$$

(46) is the equation of a straight bar with constant inertia I_c. Its first eigenvalue must be rejected, because it is relative to an extensional deformed shape, bringing us to the second eigenvalue.

For a two-hinge arch, we have

$$H_{cr} = 4\pi^2 \frac{EI_c}{l^2},$$

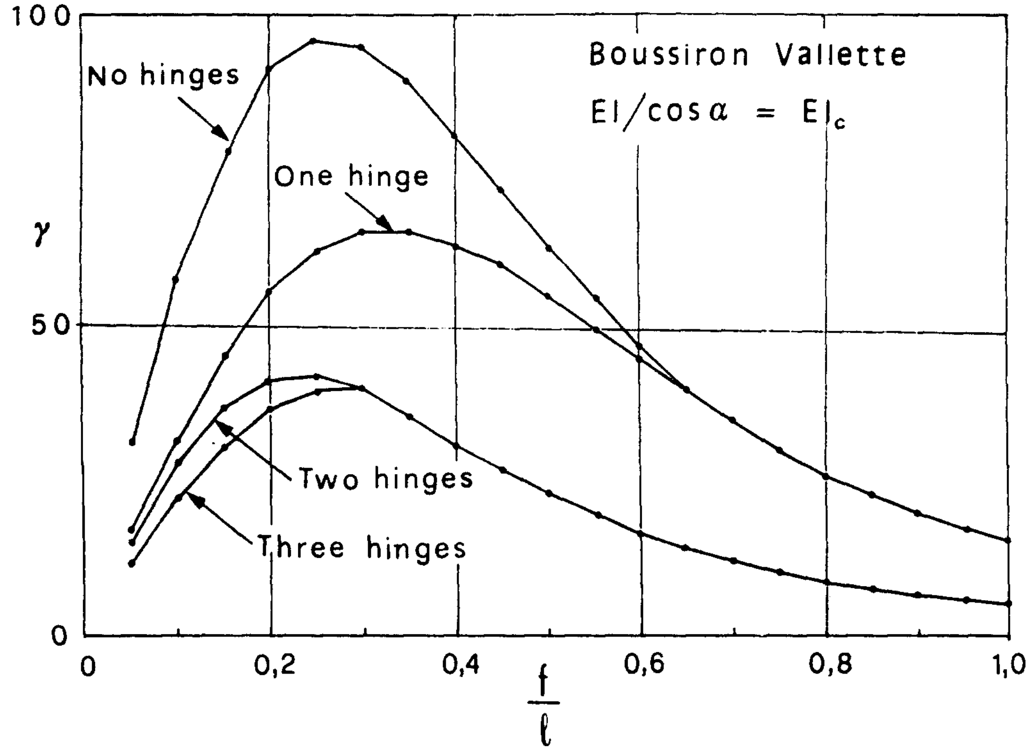

Fig. 13.

Table 3. Nondimensional coefficient γ for $EI \cos^3 \alpha$ = const.

	Clamped		One hinge		Two hinges		Three hinges	
f/l	Timoshenko and Gere [1]	Present authors	Timoshenko and Gere [1]	Present authors	Timoshenko and Gere [1]	Present authors	Timoshenko and Gere [1]	Present authors
0.1	65.5	64.4	36.5	35.4	30.7	30.8	23.8	24.1
0.2	134	134	75.8	77.1	59.8	59.3	51.2	51.4
0.3	204	209	—	127	81.1	83.6	+	+
0.4	277	289	187	188	101	106	+	+
0.5	—	370	—	258	—	126	+	+
0.6	444	451	332	335	142	143.7	+	+
0.7	—	530	—	420	—	160	+	+
0.8	587	605	497	509	170	174.8	+	+
0.9	—	677.8	—	601	—	188.4	+	+
1	700	745	697	695	193	200.7	+	+

Table 4. Nondimensional coefficient γ for $EI/\cos \alpha$ = const.

f/l	Clamped	One hinge	Two hinges	Three hinges
0.1	58.1	32.5	28.1	22.1
0.2	91.4	55.5	41.5	36.4
0.3	95	65.1	39.7	+
0.4	80.9	63.2	31.5	+
0.5	62.7	54.9	23.3	+
0.6	47	44.6	16.8	+
0.7	35	+	12.3	+
0.8	26.1	+	9	+
0.9	20	+	6.8	+
1	15.5	+	5.2	+

from which

$$q_{fz} = \frac{8f}{l^2} H_{cz} = 32\pi^2 \frac{f}{l} \frac{EI_c}{l^3} .$$

Hence, if $\frac{f}{l} = 1$, we have $\gamma = 315.82$.

For a clamped arch,

$$H_{cz} = 8\pi^2 \frac{EI_c}{l^2} .$$

Hence, $\alpha = 631.65$.

Dischinger gives the value of $\gamma = 315.9$ for a two-hinge arch and the value of $\gamma = 646$ for a clamped arch.

On reading the abacus of Pflüger, the values of $\alpha = 320$ and $\alpha = 640$ are obtained, respectively; from the abacus, the linearity between q_c and f/l appears obvious.

It seems probable that Pflüger did not consider the effect of the horizontal component w on the bending moment. This seems evident because his values apply to shallow arches, but are not correct in relation to steep arches.

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